GENERATING GENERAL OVERLAP/GROUPING FUNCTIONS VIA N-DIMENSIONAL OVERLAP/GROUPING FUNCTIONS

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1. INTRODUCTION

In the scope of fuzzy based-rule inference, implication functions play an important role in such construction, providing distinct scenarios in fuzzy systems. by exploring different classes of aggregations and fuzzy negations (MESIAR. and KOMORMIKOVA, 1997).

This research explores essential conditions guaranteeing that general overlap/grouping functions (PINHEIRO et al, 2021) can be constructed from the equivalent n-dimensional operators (QIAO and HU, 2018). In addition, constructive methods are considered to generate new implications through the concepts of general overlap/grouping functions (FODOR, 1995).

The presented results on aggregations contribute to comprehensive and flexible fuzzy inference systems by applying fuzzy implication structures (BACZYNNSKI and JAYARAM, 2010).

2. METHODOLOGY

As theoretical research on general overlap/grouping functions, this research considers the incremental study of n-dimensional overlap/grouping functions, not only as extensions of overlap/grouping functions, but investigating their properties and exploring main classes. So, the presented results can be further explored in inference relations in decision making fuzzy systems (BUSTINCE et al, 2012).

An aggregation (AF) $A:[0,1]^n \rightarrow [0,1]$ verifies the properties :

[A1] $A(0,0,...,0) = 0$ and $A(1,1,...,1) = 1$ (boundary conditions);

[A2] $A(x1,...,xn) \leq A(x'1,..., x'n)$ if $xi \leq x'i$, $\forall$ xi, x'i $\in [0,1]$ e $\forall i \in N_n$ (monotonicity).

An n-dimensional overlap function $O:[0,1]^n \rightarrow [0,1]$ is a continuous and commutative function verifying A1, A2 and the following two conditions:

(i) $\prod^{n}_{i=1} xi = 0 \Leftrightarrow O(x1,...,,xn)=0$; (ii) $\prod^{n}_{i=1} xi=1 \Leftrightarrow O(x1,...,,xn)=1$.

By relaxing the necessary conditions, a general overlap $GO:[0, 1]^n \rightarrow [0, 1]$ is given as a contínua and commutative function verifying A1, A2 and the next conditions:

(i) $\prod^{n}_{i=1} xi = 0 \Rightarrow GO(x1,...,,xn)=0$; (ii) $\prod^{n}_{i=1} xi=1 \Rightarrow GO(x1,...,,xn)=1$.

An n-dimensional grouping function $G:[0,1]^n \rightarrow [0,1]$ is continuous, commutative and also verifies A1, A2 and the following two conditions:

(i) $\prod^{n}_{i=1} xi = 0 \Leftrightarrow G(x1,...,,xn)=0$; (ii) $\prod^{n}_{i=1} xi=1 \Leftrightarrow G(x1,...,,xn)=1$.

By relaxing the necessary conditions, a general grouping $GG:[0, 1]^n \rightarrow [0, 1]$ is continuous, commutative satisfying A1 and A2 together with the conditions

(i) $\prod^{n}_{i=1} xi = 0 \Rightarrow GG(x1,...,,xn)=0$; (ii) $\prod^{n}_{i=1} xi=1 \Rightarrow GG(x1,...,,xn)=1$.

See Table 1, illustrating general overlap/grouping (O/G) functions and n-dimensional overlap/grouping (O_n/G_n), including related bivariate expressions.

Table 1. General-overlap and General-grouping functions.

General Overlap Function $\mathcal{O}$	O	O_n
$\mathcal{O}_{mM}(\vec{x}) = \min_{i=1}^{n} x_i \cdot \max_{i=1}^{n} x_i^p,\ p > 0$	✓	✓
$\mathcal{O}_{EP}(\vec{x}) = \frac{\prod_{i=1}^{n} x_i}{1+\prod_{i=1}^{n}(1-x_i)}$	✓	✓
$\mathcal{O}_S(\vec{x}) = \sin(\frac{\pi}{2}(\prod_{i=1}^{n} x_i)^p), p > 0$	✓	✓
$\mathcal{O}_M(\vec{x}) = \min_{i=1}^{n} x_i^p,\ p > 0$	✓	✓
$\mathcal{O}_L(\vec{x}) = \max((\Sigma_{i=1}^{n} x_i) - (n-1), 0)$	✗	✗
$\mathcal{O}_U(\vec{x}) = \begin{cases} n\prod_{i=1}^{n} x_i, \text{ if } \prod_{i=1}^{n} x_i \leq \frac{1}{n}, \\ 1, \text{otherwise.} \end{cases}$	✗	✗
$\mathcal{O}_G(\vec{x}) = \begin{cases} nO_L(\vec{x}), & \text{if } O_L(\vec{x}) \leq \frac{1}{n}, \\ 1, & \text{otherwise.} \end{cases}$	✗	✗

General Grouping Function $\mathcal{G}$	G	$\mathcal{G}_n$
$\mathcal{G}_{EP}(\vec{x}) = \frac{\sum_{i=1}^{n} x_i}{1+\prod_{i=1}^{n} x_i}$	✓	✓
$\mathcal{G}_O(\vec{x}) = \max_{i=1}^{n} x_i^p,\ p > 0$	✓	✓
$\mathcal{G}_L(\vec{x}) = \left(1 - \prod_{i=1}^{n}(1-x_i)\right)\min(\sum_{i=1}^{n} x_i, 1)$	✓	✓
$\mathcal{G}_{ML}(\vec{x}) = \left(1 - \sqrt[n]{\prod_{i=1}^{n}(1-x_i)}\right)\min(\sum_{i=1}^{n} x_i, 1)$	✓	✓
$\mathcal{G}_{LK}(\vec{x}) = \min((\Sigma_{i=1}^{n} x_i), 1)$	✗	✗
$\mathcal{G}_B(\vec{x}) = \min(1, n - \sum_{i=1}^{n}(1-x_i)^2)$	✗	✗
$\mathcal{G}_k(\vec{x}) = \begin{cases} 0, & \text{if } \max_{i=1}^{n} x_i \leq k \\ \frac{1}{1-k}(\max_{i=1}^{n} x_i - k), & 0 \leq k \leq 1 \end{cases}$	✗	✗

3. RESULTS AND DISCUSSION

This section introduces the methods to generate GOF/GGF based on an aggregation A:$[0,1]^n \to [0,1]$. For $0 \leq a < b \leq 1$, the function $A_a^b:[0,1]^n \to [0,1]$ given as

$$A_a^b(\vec{x}) = \begin{cases} 0, & \text{if } A(\vec{x}) \leq a \\ 1, & \text{if } A(\vec{x}) \geq b \\ \frac{A(\vec{x})-a}{b-a}, & \text{if } a < A(\vec{x}) < b. \end{cases}$$

One can easily observe that, when b = 1, $A_a = A_a^1$ and a = 0, $A^b = A_0^b$. Now, we show the conditions under which GOF can be obtained from AF.

Proposition 1. Let $0 \leq a < b \leq 1$ and $O_n:[0,1]^n \to [0,1]$ be an n-DOF. Then,

(a) O_a^b is a GOF which satisfies neither (i) nor (ii);

(b) O_a is a GOF not satisfying (i) but holding (ii); and

(c) O_b is a GOF holding (i) but not satisfying (ii).

Corollary 1.1: Let $a = \frac{1}{4}$, $b = \frac{3}{4}$ and the bivariate n-DOF $O_m:[0,1]^2 \to [0,1]$ given as $O_m(x,y) = min(\sqrt{x}, \sqrt{y})$. By Prop. 1, GOF generated from O_m is illustrated:

$$(\mathcal{O}_m)^{\frac{3}{4}}(x,y)=\begin{cases} \frac{4}{3}\sqrt{x}, \text{ if } x \leq y \text{ and } x \leq \frac{9}{16} \\ \frac{4}{3}\sqrt{y}, \text{ if } y < x \text{ and } y < \frac{9}{16} \\ 1, \text{ if } x > \frac{9}{16} \text{ and } y > \frac{9}{16}. \end{cases}$$

$$(\mathcal{O}_m)^{\frac{3}{4}}_{\frac{1}{4}}(x,y)=\begin{cases} 0, \text{ if } \min(x,y) \leq \frac{1}{16} \\ 1, \text{ if } \min(x,y) \geq \frac{9}{16} \\ 2(\sqrt{x} - \frac{1}{4}), \text{ if } x \leq y \text{ and } \frac{1}{16} < x < \frac{9}{16} \\ 2(\sqrt{y} - \frac{1}{4}), \text{ if } y \leq x \text{ and } \frac{1}{16} < y < \frac{9}{16}. \end{cases}$$

$$(\mathcal{O}_m)_{\frac{1}{4}}(x,y)=\begin{cases} 0, \text{ if } \min(x,y) < \frac{1}{16} \\ \frac{4}{3}(\sqrt{x} - \frac{1}{4}), \text{ if } \frac{1}{16} \leq x \leq y \\ \frac{4}{3}(\sqrt{y} - \frac{1}{4}), \text{ if } \frac{1}{16} \leq y \leq x. \end{cases}$$

Corollary 1.2: Take $a = \frac{1}{4}$, $b = \frac{\sqrt{3}}{4}$ and S:$[0,1]^2 \to [0,1]$, $O_S(x,y)=\sin(\frac{\pi}{4}xy)$ as a bivariate overlap function. By Prop.1, see other GOF given as:

$$(\mathcal{O}_S)^{\frac{\sqrt{3}}{2}}(x,y)=\begin{cases}\frac{2\sqrt{3}}{3}\sin(\frac{\pi}{2}xy), & \text{if } 0 \leq xy \leq \frac{2}{3}\\ 1, & \text{otherwise.}\end{cases}$$

$$(\mathcal{O}_S)^{\frac{\sqrt{3}}{2}}_{\frac{1}{2}}(x,y)=\begin{cases}0, & \text{if } 0 \leq xy \leq \frac{1}{3}\\ 1, & \text{if } \frac{2}{3} \leq xy \leq 1\\ \frac{2\sin(\frac{\pi}{2}xy)-1}{\sqrt{3}-1}, & \text{otherwise.}\end{cases}$$

$$(\mathcal{O}_S)_{\frac{1}{2}}(x,y)=\begin{cases}0, & \text{if } 0 \leq xy \leq \frac{1}{3}\\ 2\sin(\frac{\pi}{2}xy)-\frac{1}{2}, & \text{otherwise.}\end{cases}$$

Proposition 2. Let $G_n : [0,1]^n \rightarrow [0,1]$ be an n-DGF. Then, the following holds:

(a) G_a^b is a GGF that satisfies neither (i) nor (ii).
(b) G_a is a GGF not satisfying (i) but holding (ii).
(c) G^b is a GGF holding (i) but not satisfying (ii).

Corollary 2.1. Let $G: [0,1]^2 \rightarrow [0,1]$, $G_m(x,y) = min(\sqrt{1-x}, \sqrt{1-y})$ be a bivariate grouping function. By Prop. 2, see the related GGF:

$$(\mathcal{G})^{\frac{3}{4}}(x,y)=\begin{cases}0, & \text{if } x=y=0\\ 1, & \text{if } x,y \in [\frac{15}{16},1]\\ \frac{4}{3}(1-\min(\sqrt{1-x},\sqrt{1-y}), & \text{otherwise.}\end{cases}$$

$$(\mathcal{G})^{\frac{3}{4}}_{\frac{1}{4}}(x,y)=\begin{cases}0, & \text{if } x,y \in [0,\frac{7}{16}]\\ 1, & \text{if } x,y \in [\frac{15}{16},1]\\ \frac{3}{2}-2\sqrt{1-x}, & \text{if } x \geq y \text{ and } \frac{7}{16} < x < \frac{15}{16}\\ \frac{3}{2}-2\sqrt{1-y}, & \text{if } y \geq x \text{ and } \frac{7}{16} < y < \frac{15}{16}.\end{cases}$$

$$(\mathcal{G})_{\frac{1}{4}}(x,y)=\begin{cases}0, & \text{if } x,y \in [0,\frac{7}{16}]\\ 1, & \text{if } x=1 \text{ or } y=1\\ 1-\frac{4}{3}\sqrt{1-x}, & \text{if } x \geq y \text{ and } \frac{7}{16} < x < 1\\ 1-\frac{4}{3}\sqrt{1-y}, & \text{if } y \geq x \text{ and } \frac{7}{16} < y < 1.\end{cases}$$

Corollary 2.2. Let $G_B:[0,1]^n \rightarrow [0,1]$, $G_B(x,y)=\min(1,2-(1-x)^2-(1-y)^2$ be a bivariate grouping function, $a = \frac{1}{4}$ and $b = \frac{3}{4}$. By Prop. 2, G_B generates the following GGF:

$$(\mathcal{G}_B)^{\frac{3}{4}}(x,y)=\begin{cases}0, & \text{if } x=y=1\\ 1, & \text{if } 0 \leq \min(\sqrt{1-x},\sqrt{1-y}) \leq \frac{1}{4}\\ \frac{4}{3}(1-\min(\sqrt{1-x},\sqrt{1-y}), & \text{otherwise.}\end{cases}$$

$$(\mathcal{G}_B)^{\frac{3}{4}}_{\frac{1}{4}}(x,y)=\begin{cases}0, & \text{if } \frac{7}{4} \leq (1-x)^2+(1-y)^2 \leq 2\\ 1, & \text{if } 0 \leq (1-x)^2+(1-y)^2 \leq \frac{5}{4}\\ \frac{7}{2}-2((1-x)^2+(1-y)^2), & \text{otherwise.}\end{cases}$$

$$(\mathcal{G}_B)_{\frac{1}{4}}(x,y)=\begin{cases}0, & \text{if } \frac{7}{4} \leq (1-x)^2+(1-y)^2 \leq 2\\ 1, & \text{if } x=1 \text{ or } y=1\\ \frac{4}{3}\min(\frac{2}{3},\frac{5}{3}-(1-x)^2-(1-y)^2), & \text{otherwise.}\end{cases}$$

4. CONCLUSIONS

The proposal recovers the characteristics of general overlap/grouping functions, proposing a constructive model for the generation of general overlap (grouping) functions by n-dimensional overlap (grouping) functions. Examples of classes are presented in order to validate the methods.

Further works include the generalization of other properties such as the O-conditionality law and distributivity laws of implications (DIMURO et al, 2029). So, the results can be extended for the interval-valued approach by considering admissible orders (CAO and HU, 2021).

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